[N294] Analysis of Tire Vibration by Using a Hybrid Two-Dimensional Finite Element Based on Composite Shell Theory

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ABSTRACT

It has been shown experimentally that the vibration of a tire in the region near the contact patch can be represented by a set of decaying waves, each associated with a particular cross-sectional mode. Thus, a tire can be modeled as a lossy waveguide in which decaying waves propagate in the circumferential direction. It may therefore be computationally efficient to analyze tire vibration, especially in the region close to the contact patch, by using a hybrid finite element model in which the cross-section of a tire is approximated by two-dimensional finite elements while a wave-like solution is assumed in the circumferential direction. Here, a hybrid finite element was formulated based on composite shell theory: in particular, a circular conical shell element was formulated. The inflation pressure acting on the inner surface was included in the model by considering both residual stresses and non-linear terms in the strain-displacement relations. The dispersion relations for the tire model obtained by using the hybrid FE model were compared with those obtained from a full, three-dimensional FE model. It has been shown that the FE analysis made using the hybrid two-dimensional finite elements yields results in close agreement with a three-dimensional model.

KEYWORDS: tire, hybrid finite element, dispersion relation, waveguide
INTRODUCTION

In earlier experimental work, a stationary tire was driven radially at a point on its treadband and measurements of the resulting radial treadband vibration were made around the treadband circumference by using a laser Doppler velocimeter. By performing a circumferential wave number transform of the measured space-frequency data, the wave propagation characteristics of a tire could be visualized [1]. In an attempt to understand these experimental results in detail, the tire treadband was modeled as a circular cylindrical shell with air pressure acting on its interior surface [2,3]. To identify effects of tire rotation on wave propagation, the rotation of a circular cylindrical shell was also considered [3]. The shell model was found to be capable of representing the principal wave propagation characteristics of a tire: i.e., the vibrational response of a tire can be expressed as a superposition of decaying waves, each associated with a particular cross-sectional mode shape. Thus, it was concluded that a tire can be modeled as a lossy waveguide [1-3].

When a FE model of a tire is used to analyze tire vibration at high frequencies, the size of the elements must be small and the tire’s cross-section should be modeled in detail since the vibrational wavelengths may be comparable to the thickness of the tire. In consequence, a full, 3-D finite element model for high frequency analysis may require both a large modeling effort and significant computational resources.

Since tires behave like constant cross-section waveguides, it would be computationally efficient to analyze tire vibration by using hybrid, 2-D FE models: i.e., the cross-section of a tire is approximated by finite elements while a wave-like solution is assumed in the circumferential direction. Note that the hybrid, 2-D FE models of the type to be described here can also be used to investigate the dynamic behavior of any structure whose cross-sectional shape and material properties can be assumed to be constant with respect to the circumferential direction (e.g., tire, disc, or bell) or the axial direction (e.g., plate, straight ventilation duct, or aircraft fuselage).

Previously, Cheung [4] described a hybrid, 2-D FE formulation based on the use of strip elements: interpolation functions were prescribed in the cross-sectional direction along with analytical mode shapes in the axial direction. Richards [5] analyzed the vibrational response of a tire coupled with an internal acoustical cavity by applying hybrid 2-D finite elements to both the tire and acoustical cavity. In his case, the tire was modeled as a membrane: i.e., treadband flexural stiffness was not accounted for. In addition, Brockman et al. [6] estimated tire critical speeds by using a hybrid 2-D FE model. They used solid elements in cylindrical coordinates and accounted for the tire’s rotation in the circumferential direction; inflation pressure was also considered by including initial stresses and non-linear strains in their formulation. Nilsson and Finnveden [7] calculated the input point mobility of a tire by using a hybrid 2-D FE model based on orthotropic, pre-stressed conical shell elements.

Here, a hybrid 2-D finite element for a circular, conical shell is described. When the initial static stresses in a shell element are assumed to be much larger than the dynamic stresses associated with vibration, the non-linear strain energy terms represented by the multiplication of initial stresses and non-linear strains cannot be neglected [6,8]. Thus, the hybrid, 2-D finite shell element presented here includes non-linear strains to accommodate the latter situation. In addition, multi-layered, thin shell elements (referred to as composite shell elements) with constant thickness in the cross-sectional direction were also incorporated. Finally, allowance was made for both external point forces applied at a node and distributed forces exerted on an element.

As a first step in the application of the hybrid 2-D finite element, a tire was modeled by using orthotropic, circular conical shell finite elements. A full, 3-D FE model that had the same geometry and material properties as the hybrid 2-D model was also analyzed for the purpose of comparison.
FINITE ELEMENT FORMULATION

A. Strain-Displacement Relations

Figure 1 shows a sketch of a circular conical shell finite element that has two nodal lines. Here we define the local element coordinates, \( x_1 \) as the cross-sectional direction, \( x_2 \) as the circumferential direction, and \( x_3 \) as the normal direction to the shell surface. It is assumed that vibrational displacements of the shell element can be approximated by interpolation functions in the \( x_1 \)-direction and represented by an analytical solution in the \( x_2 \)-direction. Then, the displacement vector \( \mathbf{u} = [u_1, u_2, u_3]^T \) of the element can be expressed as

\[
\mathbf{u}(x_1, x_2, t) = \mathbf{\chi}(x_1) \mathbf{y}(t, x_2),
\]

where \( \mathbf{\chi} \) is the matrix of interpolation functions, \( \mathbf{y} \) is the nodal displacement vector, and \( t \) is the time. Note that an element has two nodes and that each node has four nodal displacements (three translational displacements and one rotational displacement) that are functions of \( t \) and \( x_2 \).

When the shear deformation of the shell is assumed to be negligible, strain can be separated into membrane and bending strains, and those strains can be directly related to displacements (see Ref. [8]). By substituting Eq. (3) into the strain-displacement relations, strains can be associated with the nodal displacement vector: i.e.,

\[
\mathbf{e}(x_1, x_2, t) = \mathbf{E}_0(x_1) \mathbf{y}(x_2, t) + \mathbf{E}_1(x_1) \frac{\partial \mathbf{y}(x_2, t)}{\partial x_2} + \mathbf{E}_2(x_1) \frac{\partial^2 \mathbf{y}(x_2, t)}{\partial x_2^2} + \cdots,
\]

where \( \mathbf{e} \) is the strain vector, \( \mathbf{e} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \kappa_{11}, \kappa_{22}, \kappa_{12}]^T \), \( \epsilon_{\text{mem}} \) is the membrane strain, and \( \kappa_{\text{cur}} \) is the bending strain. Note that in Eq. (4), the strain vector is separated in terms of independent variables: i.e., the matrices, \( \mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2 \), which are functions of \( x_1 \), and the derivatives of the nodal displacements which are functions of \( x_2 \) and \( t \). Note also that the strain vector can be separated into two parts, i.e., \( \mathbf{e} = \mathbf{e}_l + \mathbf{e}_n \), where the first part represents the linear strain vector and the second the non-linear strain vector: the first three terms on the right-hand side of Eq. (4) represent the linear strain-displacement relations.

B. Energy Expressions

In a composite shell, the resultant forces obtained by integrating the stresses in the thickness direction (i.e., the \( x_3 \)-direction) can be related to the strains by

\[
\mathbf{R} = \mathbf{C} \mathbf{e},
\]

where \( \mathbf{R} = [N_{11} N_{22} N_{12} M_{11} M_{22} M_{12}]^T \): i.e., the resultant force vector. When there are initial stresses, the resultant forces can be expressed as the sum of the initial resultant forces and the dynamic resultant forces: i.e., \( \mathbf{R} = \mathbf{R}_o + \mathbf{R}' \), where the first term, \( \mathbf{R}_o \), is the resultant force vector which results from the initial stresses, and the second term, \( \mathbf{R}' \), is the dynamic resultant force vector which is associated with the shell motion.

When the non-linear strains are much smaller than the linear strains (\( \epsilon_l \gg \epsilon_n \)) and the initial resultant forces are much larger than the dynamic resultant forces (\( \mathbf{R}_o \gg \mathbf{R}' \)), the potential energy can be approximated as

\[
U_l = \frac{1}{2} \int \int \mathbf{e}_l^T \mathbf{C}_l \mathbf{A}_l \mathbf{A}_l \mathbf{dx}_1 \mathbf{dx}_2 dt
\]

and

\[
U_n = \frac{1}{2} \int \int \mathbf{e}_n^T \mathbf{R}_o \mathbf{A}_l \mathbf{dx}_1 \mathbf{dx}_2 dt,
\]

where \( \mathbf{A}_l \) and \( \mathbf{A}_n \) are Lamé parameters [8]. The kinetic energy can be expressed as

\[
T = \frac{1}{2} \int \int \rho \mathbf{v} \dot{\mathbf{v}} \mathbf{\chi} \mathbf{\chi} \mathbf{A}_l \mathbf{A}_l \mathbf{dx}_1 \mathbf{dx}_2 dt,
\]
where \( \rho \) is the density. Finally, the work performed by external distributed forces is
\[
W_q = \iint y^T \chi^T Q A_2 dx_1 dx_2 dt
\tag{9}
\]
and the work performed by external point forces is
\[
W_f = \iint y^T \chi^T F A_2 dx_1 dx_2 dt
\tag{10}
\]
where \( Q \) and \( F \) are the external distributed and point force vectors, respectively.

**C. Element Equations**

The element equations can be obtained by taking a small variation of the nodal displacements in the energy expression: i.e.,
\[
\delta \left( U_j + U_m - T - W_q - W_f \right) = 0.
\tag{11}
\]
By substituting Eqs. (6) to (10) into Eq. (11), the system equation for one element can be derived as
\[
\sum_{m=0}^{4} K_m \frac{\partial^2 y}{\partial x_m^2} + \sum_{m=0}^{2} K_0 \frac{\partial^2 y}{\partial x^m} + M \frac{\partial^2 y}{\partial t^2} = F' + F + Q',
\tag{12}
\]
where \( K_m \) is the 8 by 8 \( m \)-th stiffness matrix (the superscript, “0” denotes that the stiffness matrix is generated by initial stresses), \( M \) is the 8 by 8 mass matrix, \( F' \) and \( F \) are the 8 by 1 internal and external (\( F' = [F_{11} F_{12} F_{21} F_{22} F_{31} F_{32} F_{41} F_{42}]^T \)) nodal force vectors, respectively, and \( Q' \) is the 8 by 1 distributed force vector (\( Q' = [Q_{11} Q_{12} Q_{21} Q_{22} Q_{31} Q_{41} Q_{42}]^T \)).

When more than one element is used to represent a system, a global system equation can be assembled from the individual element equations by applying conditions of displacement continuity and force balance at each node. Note that during the latter procedure, the internal force vectors that appear in Eq. (12) cancel out.

**D. Solution Procedure**

Once the global system equation is obtained, boundary conditions should be applied. To obtain initial stresses, the static equation, which does not include the mass matrix and the stiffness matrices associated with the initial stresses, should first be solved under the appropriate static forcing condition to yield the static displacements. Then the resulting static displacements can be used to obtain the initial resultant forces by using Eq. (5) combined with Eq. (4). Based on those forces, the stiffness matrices associated with the initial stresses can be calculated.

In a dynamic analysis, the natural frequencies and modes are first obtained from the global system equation without external forces. Note that the natural vibration response must satisfy the condition of circular symmetry. Thus, the natural modes can be represented as
\[
\mathbf{u}_{mm}(x_1, x_2, t) = \mathbf{U}_{mm}(x_1) \exp(-inx_2 + i\omega_{mn} t),
\tag{13}
\]
where \( x_2 \) is the circumferential direction and \( n \) is the circumferential mode number, which can be an arbitrary integer. By substituting Eq. (13) into the global system equation, an eigenvalue problem is formulated. The harmonic, forced response can then be represented by modal superposition as
\[
\mathbf{u}(x_1, x_2, t) = \sum_m \sum_n \eta_{mn} \mathbf{U}_{mn} \exp(-inx_2 + i\omega t),
\tag{14}
\]
where the modal coefficient, \( \eta_{mn} \), is
\[
\eta_{mn} = \frac{f_{mn}}{-\omega^2 + \omega_{mn}^2 + i2\omega\xi_{mn}},
\tag{15}
\]
\( \xi \) is the modal damping ratio.
\[ f_{mn} = \frac{1}{\lambda_{mn}} \int_0^{2\pi} \mathbf{U}_{mn}^H \left( \mathbf{F}^e + \mathbf{Q}^e \right) \exp(inx_2) dx_2, \]  
(16)

and

\[ \lambda_{mn} = \mathbf{U}_{mn}^H \overline{\mathbf{M}}_{mn} \mathbf{U}_{mn}, \]  
(17)

where the upper bar on the matrices indicates the global system matrices.

**TIRE MODEL**

Figure 2 shows the cross-sectional geometry of a tire. The cross-sectional center points and thicknesses of an uninflated tire were measured at 42 points across a cross-section. Then the two sets of measured data were curve-fitted and 37 points were re-sampled from the resulting curves as shown in Fig. 2. Note that nodes 12 to 26 were used to define the treadband elements and nodes 1 to 12 and 26 to 37 were used to define the sidewall elements. Note also that the treadband thickness is assumed to be constant (see Fig. 2(b)).

Based on the cross-sectional geometry, the tire was modeled by using both hybrid, 2-D finite elements and full, 3-D finite elements: i.e., the hybrid, 2-D FE model consisted of 36 elements (37 nodes) while 36x90 elements (90 elements around the half circle) were used for the full, 3-D FE model (see Fig. 3). Note that the full, 3-D FE model was implemented in ANSYS Version 6.0 and element type SHELL63 was used. Note also that only the upper half tire was modeled in the full, 3-D FE model since symmetric boundary conditions were applied. Different sets of orthotropic material properties were used for the treadband and sidewall as shown in Table 1: they were adapted from the literature [9], were based on physical reasoning, or were obtained by direct measurement of tires. All translational displacements at the edges of sidewalls (i.e., at the bead) were constrained to be zero. An inflation pressure of 207 kPa (20 psi) was applied to the inside surface for the static analysis, and a point force at the center of the treadband was applied for the dynamic analysis.

**RESULTS AND DISCUSSION**

The natural frequencies, \( f_{mn} = \frac{\omega_{mn}}{2\pi} \), obtained by using the hybrid, 2-D FE model are shown in Fig. 4 as the function of circumferential mode number, \( n \): Fig. 4(a) shows the results without inflation pressure while Fig. 4(b) is for the case with inflation pressure. Note that each point in the frequency-circumferential mode number plane is associated with a particular wave type and cross-sectional mode shape [1,2]. Thus, the first index, \( m \), in the natural frequency, \( f_{mn} \), denotes both the wave type and cross-sectional mode shape, while the second index, \( n \), denotes the circumferential mode number. It can be seen that when inflation pressure is applied, the first flexural wave, i.e., the trajectory having the lowest natural frequencies for all circumferential mode numbers, behaves like a membrane wave: i.e., the points associated with this wave type “straighten out” when the inflation pressure is applied [1,2].

Forced responses in the frequency-circumferential mode number domain are shown in Figs. 5 and 6: the results in Fig. 5 were obtained by using the full, 3-D FE model while the results in Fig. 6 were obtained by using the hybrid, 2-D FE model. To obtain these results, vibrational velocities around the tire circumference at the treadband center points were first calculated by using the FE models and the resulting velocity data, represented in the spatial domain at each frequency, were then decomposed into wave number components, \( \kappa \) = \( n/R \), where \( R \) is the radius of the tire circumference, by using spatial Fourier transforms. Note that the circumferential phase speed of a particular trajectory is represented by the ratio of angular natural frequency to circumferential wave number (\( \omega_{mn}/\kappa \)). By comparing Figs. 5(a) and 5(b) (or Figs. 6(a) and 6(b)), the circumferential phase speeds of, in particular, flexural wave types, increase when the inflation pressure is applied [1,2]. It can also be seen
that the results obtained by using the hybrid, 2-D FE model are nearly identical to those obtained by using full, 3-D FE model (compare Figs. 5(a) and 6(a) or Figs. 5(b) and 6(b)).

CONCLUSIONS

In this article, a hybrid 2-D finite element for a composite shell was formulated by using the variational principle. For the purpose of validating the hybrid 2-D finite element, a tire was analyzed by using both the hybrid, 2-D FE and full, 3-D FE models. The FE analysis made by using the hybrid 2-D finite elements yields results in close agreement with the three-dimensional model although the hybrid 2-D FE model features a very small number of finite elements. Furthermore, since the hybrid 2-D FE model uses an exact solution in the circumferential direction, it may yield more accurate solutions.

REFERENCES


Table 1. List of material properties.

<table>
<thead>
<tr>
<th></th>
<th>Sidewall</th>
<th>Treadband</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Moduli</td>
<td>$E_1 = 1.0 \times 10^8$ Pa</td>
<td>$E_1 = 3.2 \times 10^8$ Pa</td>
</tr>
<tr>
<td></td>
<td>$E_2 = 6.0 \times 10^7$ Pa</td>
<td>$E_2 = 7.5 \times 10^8$ Pa</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>$G_{12} = 2.0 \times 10^6$ Pa</td>
<td>$G_{12} = 6.0 \times 10^8$ Pa</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>$\nu_{12} = 0.45$</td>
<td>$\nu_{12} = 0.45$</td>
</tr>
<tr>
<td>Density</td>
<td>800 kg/m$^3$</td>
<td>1200 kg/m$^3$</td>
</tr>
</tbody>
</table>

Figure 1. Sketch of circular conical shell element.
Figure 2. Cross-sectional geometry of tire: (a) center points of tire cross-section and (b) thickness.

Figure 3. Full, three-dimensional FE model of tire.

Figure 4. Natural frequencies and circumferential mode numbers obtained by using hybrid, 2-D FE model: (a) without inflation and (b) with inflation.
Figure 5. Forced response of full, 3-D FE model: (a) without inflation and (b) with inflation.

Figure 6. Forced response of hybrid, 2-D FE model: (a) without inflation and (b) with inflation.